A Computational "Rheometer" for Turbulent Flows

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Motivations

- Standard eddy diffusivity model and Boussinesq approximation in RANS equation

$$-\frac{\partial}{\partial x_j}(\overline{u_j' u_i'}) = \frac{\partial}{\partial x_j}\left[\nu_T\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right] - \frac{\partial}{\partial x_j}\frac{2}{3}k\delta_{ij}$$

  - Researchers often ask: "how to tune $\nu_T(x, y, z)$?"
  - We study: "what should the entire operator look like?"
  - Motivations: the standard model is not truly predictive for many practical flows

Simpler Question: Scalar Transport

- Instantaneous equation of passive scalar transport (Microscopic)

$$\frac{\partial c}{\partial t} + u_j \frac{\partial c}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D_{ji}\frac{\partial c}{\partial x_i}\right)$$

- Averaged equation (Macroscopic)

$$\bar{\mathcal{L}}\bar{c} = \underbrace{\left[\frac{\partial}{\partial t} + U_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\left(D_{ji}\frac{\partial}{\partial x_i}\right) + \bar{\mathcal{L}}'\right]}_{\textbf{Linear}}\bar{c} = 0$$

- Standard eddy diffusivity model

$$\bar{\mathcal{L}}'\bar{c} = \frac{\partial}{\partial x_j}(\overline{u_j' c'}) \approx -\frac{\partial}{\partial x_j}\left[D_T \frac{\partial \bar{c}}{\partial x_j}\right]$$

Macroscopic Forcing Method (MFM)

- Investigate response to forcing

Condition: $s = \bar{s}$

$$\frac{\partial c}{\partial t} + u_j \frac{\partial c}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D_{ji} \frac{\partial c}{\partial x_i}\right) + s$$

- s survives in averaging the equation

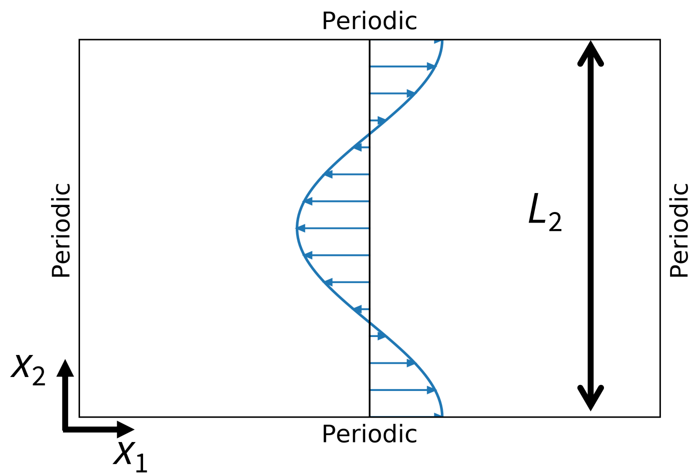
$$\bar{\mathcal{L}}\bar{c} = \left[\frac{\partial}{\partial t} + U_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}\left(D_{ji} \frac{\partial}{\partial x_i}\right) + \bar{\mathcal{L}}'\right]\bar{c} = s$$

- Perform many DNSs to compute $\bar{c}$ in response to different s
- Obtain a linear system $\bar{\mathcal{L}}\bar{c} = s$ and rearrange to obtain $\bar{\mathcal{L}}'$
- Expensive!

Example: A 2D Parallel Flow (1/6)

- Steady, parallel, velocity field

$$u_1 = U \cos\left(\frac{2\pi}{L_2} x_2\right)$$



- Example of 2D solution for c (“expensive DNS”)

- Quantity of interest $\bar{c}$

- What 1D equation (“RANS”) can directly predict $\bar{c}$?

Example: A 2D Parallel Flow (2/6)

- Dimensionless Microscopic Equation

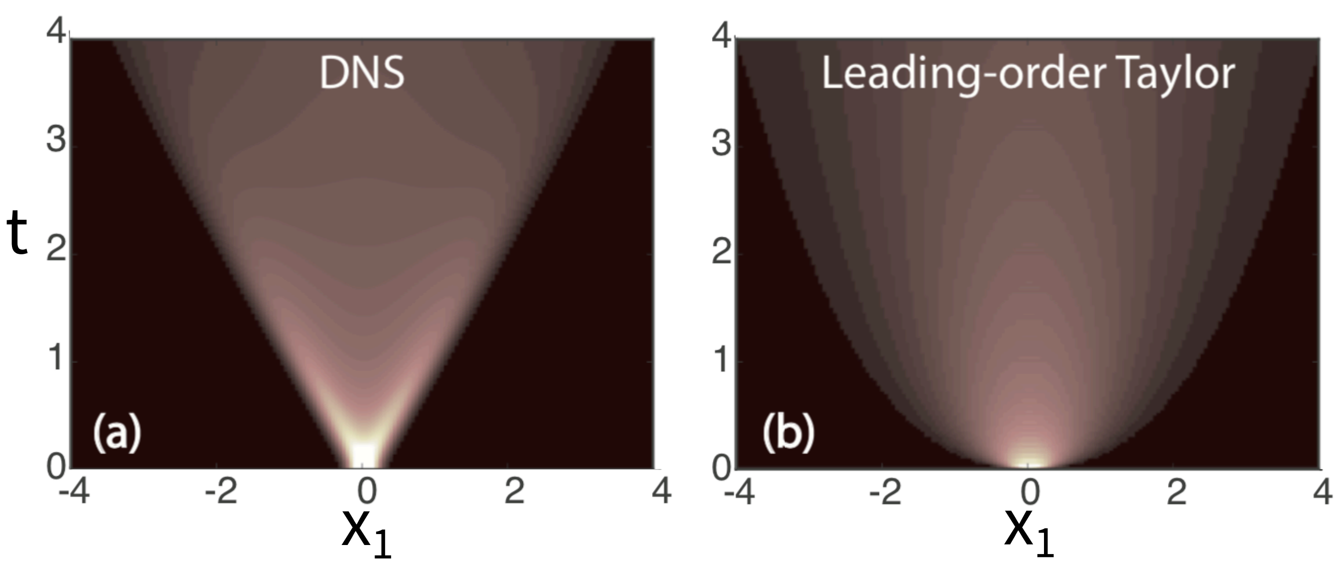
$$\frac{\partial c}{\partial t} + \cos(x_2)\frac{\partial c}{\partial x_1} = \frac{\partial^2 c}{\partial x_2^2} + \cancel{\epsilon^2 \frac{\partial^2 c}{\partial x_1^2}}, \qquad \epsilon = 2\pi D_M/(L_2 U)$$

- Approximate model using method of G. I. Taylor (1953)

$$\frac{\partial \overline{c}}{\partial t} = \frac{1}{2}\frac{\partial^2 \overline{c}}{\partial x_1^2}$$

macroscopic diffusivity = D_{eff}=1/2

valid for large-scale



Example: A 2D Parallel Flow (3/6)

- MFM analysis:
  - Add forcing to the governing equation

$$\frac{\partial c}{\partial t} + \cos\left(x_2\right)\frac{\partial c}{\partial x_1} = \frac{\partial^2 c}{\partial x_2^2} + s(x_1, t)$$

  - For different s find the linear response $\bar{c}$

  - Homogeneity allows analysis in Fourier space

$$s(x_1, t) = \exp\left(i\omega t + ikx_1\right)$$

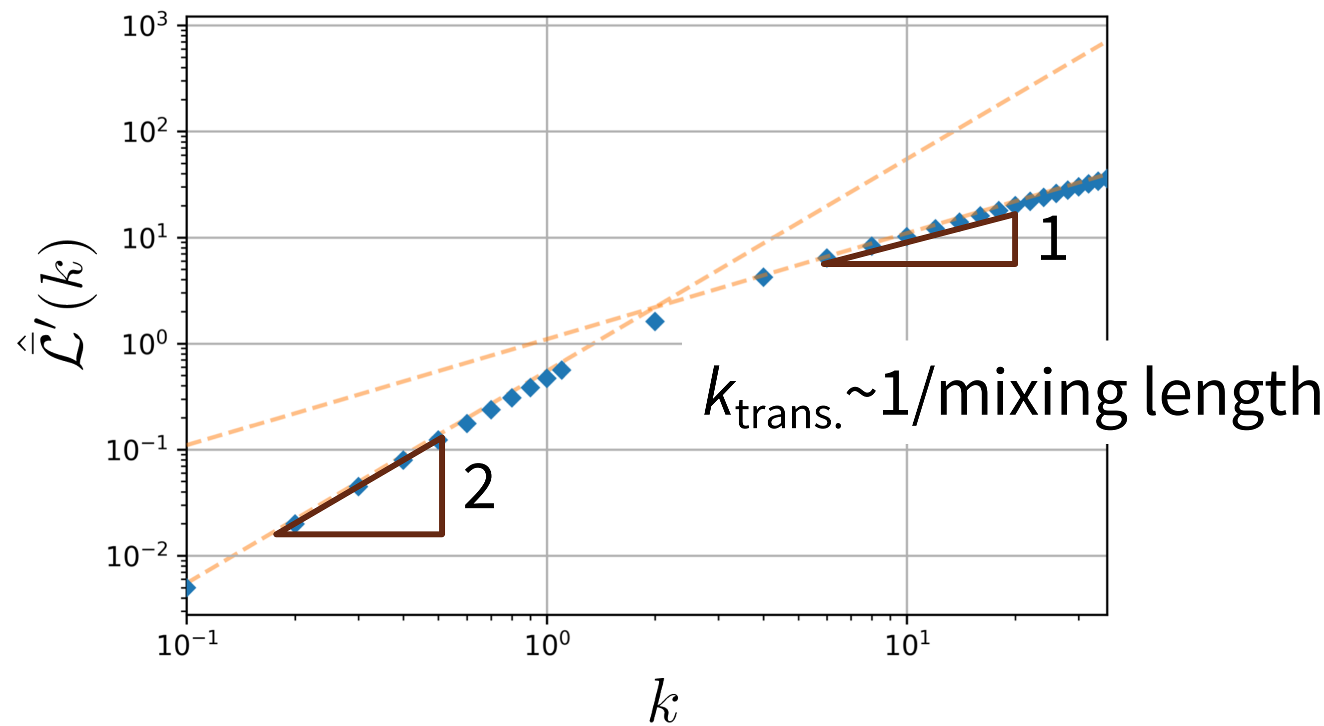
$$\Longrightarrow \bar{c}\left(x_1, t\right) = \bar{\hat{c}}\exp\left(i\omega t + ikx_1\right)$$

$$\Longrightarrow \hat{\bar{\mathcal{L}}} = 1/\bar{\hat{c}}\left(\omega, k\right)$$

Example: A 2D Parallel Flow (4/6)

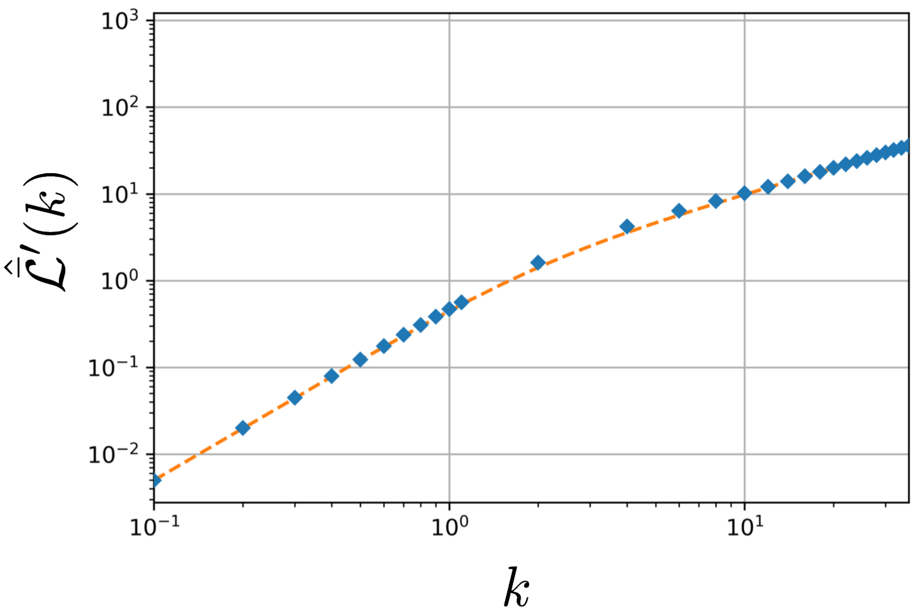
- MFM result (steady limit, ω=0)
  - Remember Taylor/Boussinesq prediction

$$\bar{\mathcal{L}}' = -D_T \frac{\partial^2}{\partial x^2} \implies \hat{\bar{\mathcal{L}}}' \propto k^2$$



Example: A 2D Parallel Flow (5/6)

- Macroscopic operator (steady limit, ω=0)



Fitted expression

$$\hat{\bar{\mathcal{L}}}'(k) = \frac{Ak^2}{\sqrt{1+(Bk)^2}}$$

- Fitted operator:

- Not a number but an operator
- Suppressed in small-scale limits
- Non-local operator

$$\boxed{\bar{\mathcal{L}}' = -\frac{\partial}{\partial x}\left[\frac{D_T}{\sqrt{1-\ell_T^2\frac{\partial^2}{\partial x^2}}}\frac{\partial}{\partial x}\right]}$$

Outline

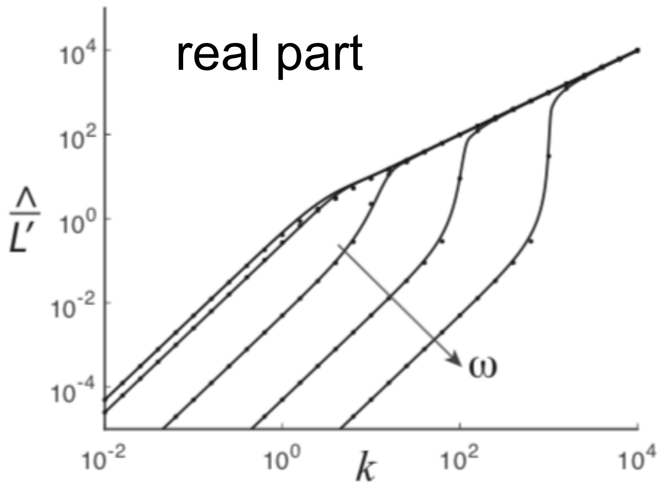
- Go back and relax simplifications

  - Unsteady problems
  - Molecular diffusivity active in all directions
  - Unsteady 3D flows
  - Extension from scalar transport to Navier-Stokes
  - Extension to non-homogeneous flows
  - Computational cost

  - An example incorporating all of the above

Example: A 2D Parallel Flow (6/6)

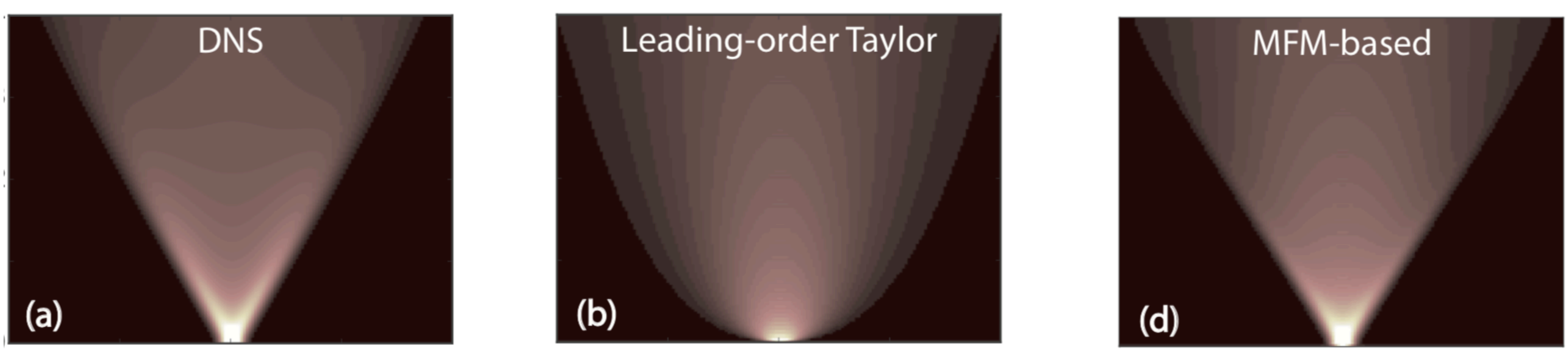
- Solution to the unsteady problem



fitted operator

$$\left[\sqrt{\left(\mathcal{I}+\frac{\partial}{\partial t}\right)^2-\frac{\partial^2}{\partial x_1^2}}-\mathcal{I}\right]\bar{c}\left(x_1,t\right)=s\left(x_1,t\right)$$

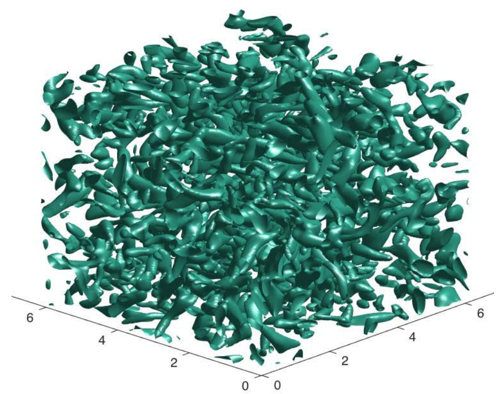
- Evaluation of performance

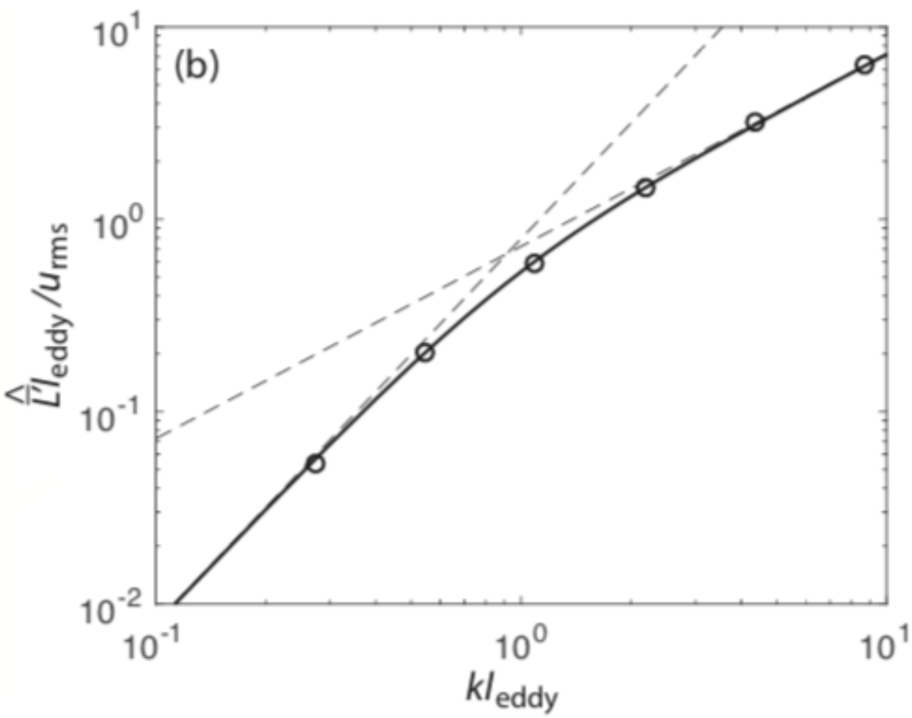


Extension to Turbulent Flows

- What will happen if velocity is 3D, unsteady, and turbulent?
- Test case: Homogeneous Isotropic Turbulence (HIT) at Re_λ=40

$$\frac{\partial c}{\partial t} + u_j \frac{\partial c}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D_{ji}\frac{\partial c}{\partial x_i}\right) + s$$





Fitted operator:

$$\boxed{\overline{\mathcal{L}'} = -\nabla . \left[\frac{D^0}{\left(\mathcal{I} - l^2\nabla^2\right)^{1/2}}\nabla\right]}$$

MFM for Navier-Stokes (1/2)

- First obtain DNS (or measurement) of flow field

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_j u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + r_i,$$

$$\frac{\partial u_j}{\partial x_j} = 0,$$

- Nonlinearity and interpretation of nonlinear term
- Apply MFM to a generalized momentum transport equation (GMT)

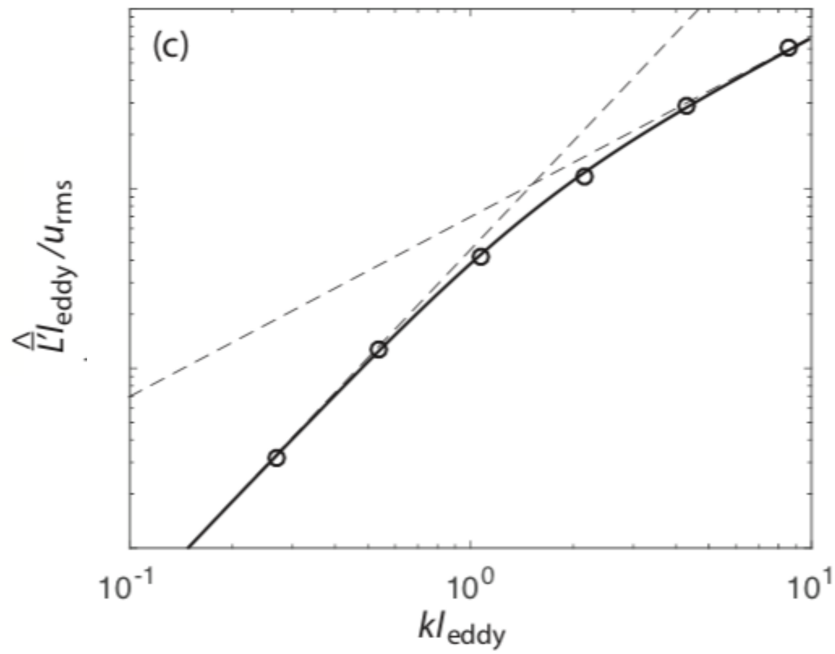
$$\frac{\partial v_i}{\partial t} + \frac{\partial u_j v_i}{\partial x_j} = -\frac{\partial q}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + s_i,$$

$$\frac{\partial v_j}{\partial x_j} = 0,$$

- Linear system, (N.S. is its special case)
- Operator dependent on flow

MFM for Navier-Stokes (2/2)

- Example: MFM + GMT applied to HIT (Re_λ=40)



Fitted operator:

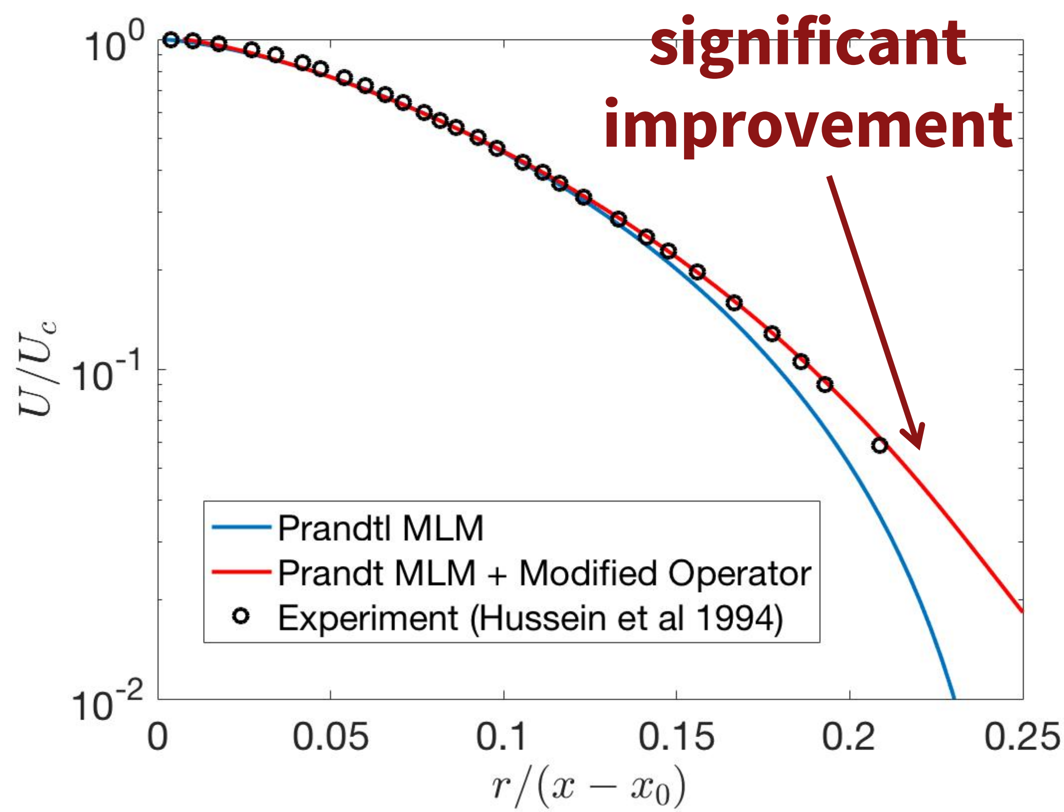
$$\boxed{\overline{\mathcal{L}'} = -\nabla . \left[\frac{D^0}{(\mathcal{I} - l^2 \nabla^2)^{1/2}} \nabla \right]}$$

- Turbulent Schmidt number $Sc_T = D^0_v / D^0_c = 0.5$

- $l_c = 1.1 l_{\text{eddy}},\ l_v = 0.6 l_{\text{eddy}}$

Impact on Prediction of Practical Flows

- Can the operator already obtained improve prediction of mean velocity profile?
  - Example: turbulent round jet → self-similar solution
  - Use Prandtl Mixing Length Model to determine *D* and *l*



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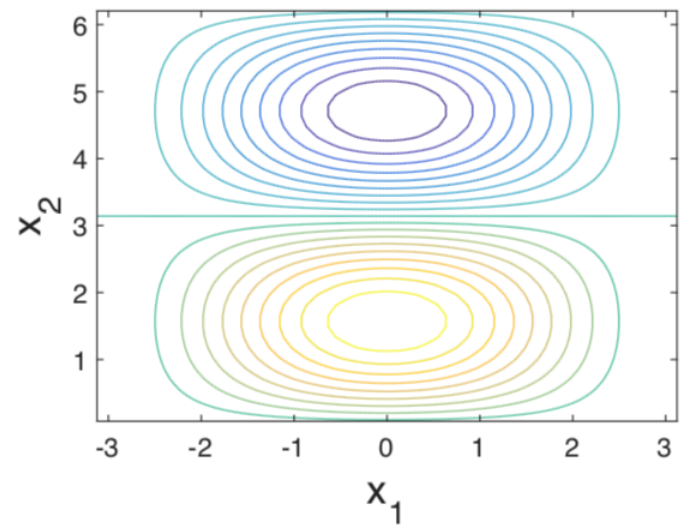
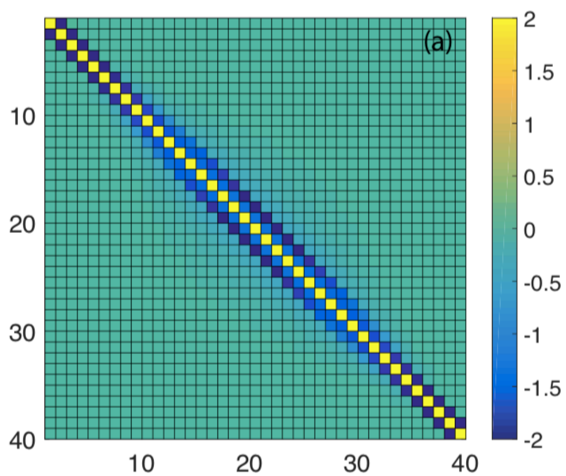
Extension to Inhomogeneous Flows (1/2)

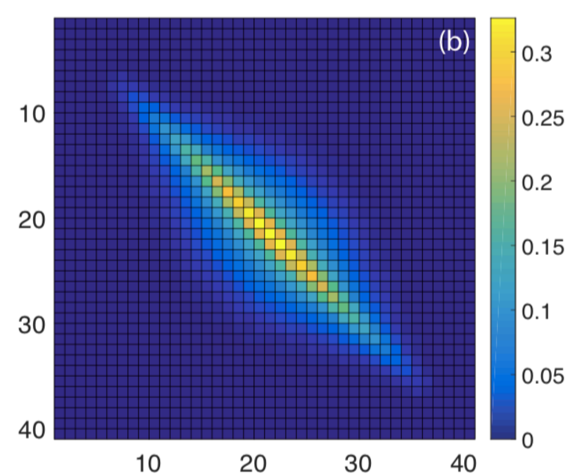
- Example: transport between two walls
  - Left/right BC: Dirichlet condition
  - Top/bottom: periodic condition
  - Averaging defined in x_2
  - Steady limit
  - Macroscopic model is 1D
- MFM Result:



$$[\overline{\mathcal{L}}] = -\left[\partial/\partial x_1\right]\left[\mathcal{D} + D_M \mathcal{I}\right]\left[\partial/\partial x_1\right]$$



$\overline{\mathcal{L}}$: discrete macroscopic operator



D = Eddy diffusivity operator
is a convolution kernel !

Most General Form of Eddy Diffusivity

- Scalar transport

$$-\overline{u_j' c'} = \mathcal{D}\nabla\overline{c} = \int_{\mathbf{y}} D_{ji}\left(\mathbf{x},\mathbf{y}\right)\frac{\partial\overline{c}}{\partial x_i}|_{\mathbf{y}}\mathbf{dy}$$

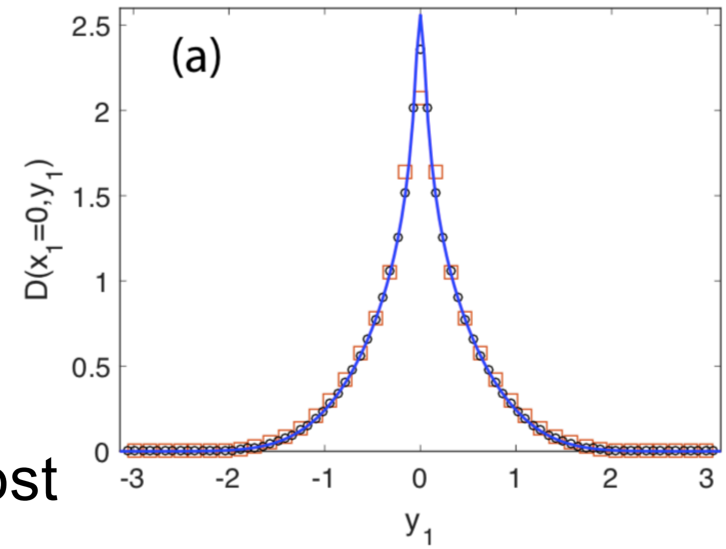
- Momentum transport

$$-\overline{u_j' v_i'} = \mathcal{D}\nabla\overline{\mathbf{v}} = \int_{\mathbf{y}} D_{jilk}\left(\mathbf{x},\mathbf{y}\right)\frac{\partial\overline{v_k}}{\partial x_l}|_{\mathbf{y}}\mathbf{dy}$$

- MFM allows precise measurement of *D*
  - Expense?

A fast method for computation of kernel moments

$$-\overline{u'c'}\left(x_1\right) = \int_{y_1} D\left(x_1, y_1\right) \frac{\partial \overline{c}}{\partial x_1}\Big|_{y_1} dy_1$$



Inverse MFM (not presented)

Can obtain moments of D at affordable cost

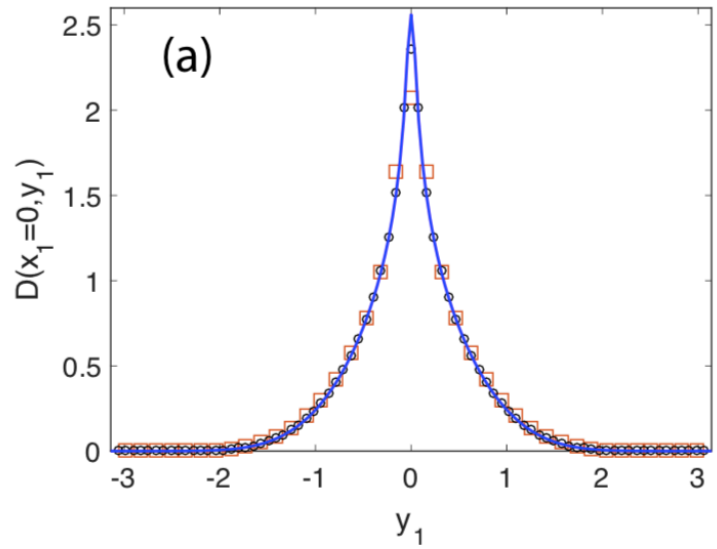
$$D^0(x_1) = \int_{y_1} D\left(x_1, y_1\right) dy_1$$

$$D^1(x_1) = \int_{y_1} \left(y_1 - x_1\right) D\left(x_1, y_1\right) dy_1$$

$$D^2(x_1) = \int_{y_1} \frac{1}{2}\left(y_1 - x_1\right)^2 D\left(x_1, y_1\right) dy_1$$

Boussinesq Approximation

$$-\overline{u'c'}\,(x_1) = \int_{y_1} D\,(x_1, y_1)\,\frac{\partial \overline{c}}{\partial x_1}\Big|_{y_1} dy_1$$

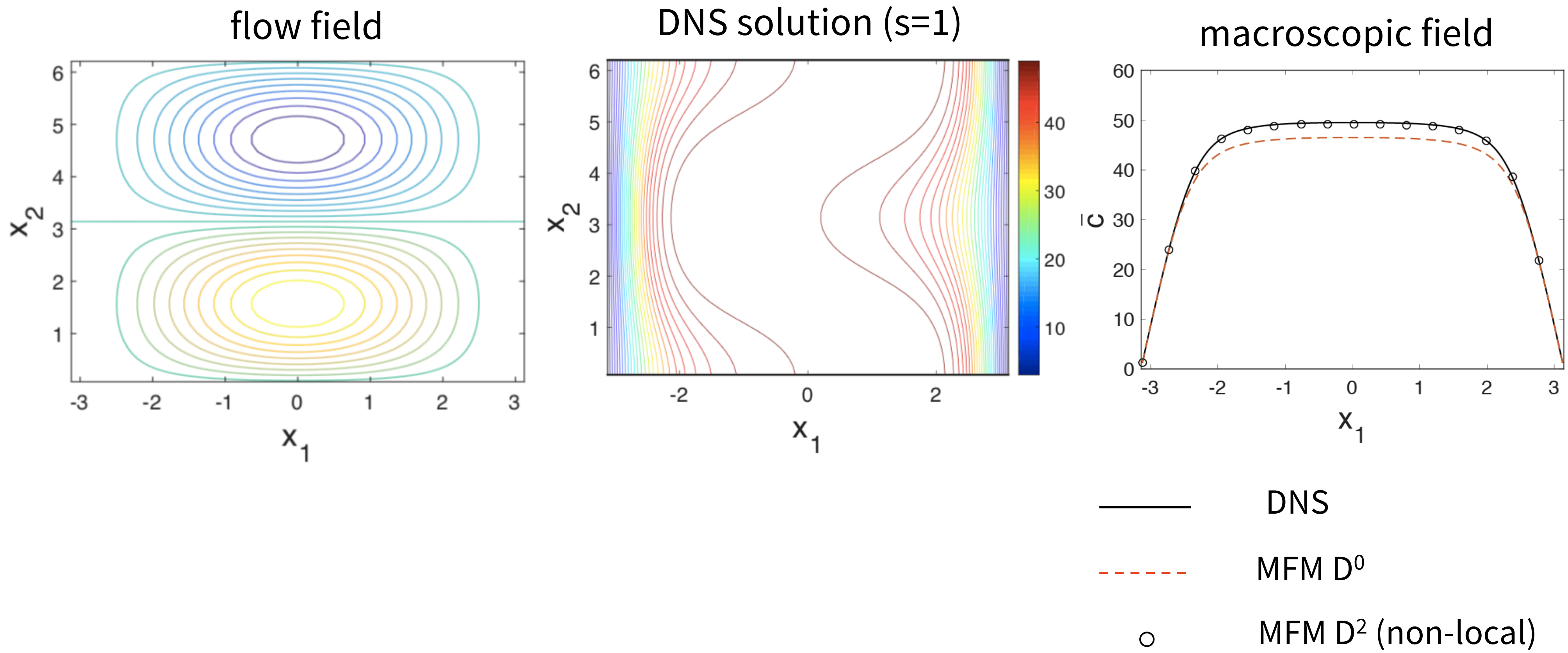


If kernel width ~ eddy size << macroscopic length

$$-\overline{u'c'}\,(x_1) \simeq \left(\int_{y_1} D\,(x_1, y_1)\,dy_1\right)\frac{\partial \overline{c}}{\partial x_1} = D^0(x_1)\frac{\partial \overline{c}}{\partial x_1}$$

➔ Provides a quantitative framework for assessment of the Boussinesq approximation

Example Application



$$\frac{\partial}{\partial x_1}\left(D_M \frac{\partial \bar{c}}{\partial x_1}\right) + \frac{\partial}{\partial x_1}\left(\int_{y_1} D\left(x_1, y_1\right) \frac{\partial \bar{c}}{\partial x_1}\Big|_{y_1} dy_1\right) = 0$$

Can construct approximate kernels by matching higher moments of $D(x_1,y_1)$

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IMFM Applied to Turbulent Channel Flow at Re_τ=180 (1/3)

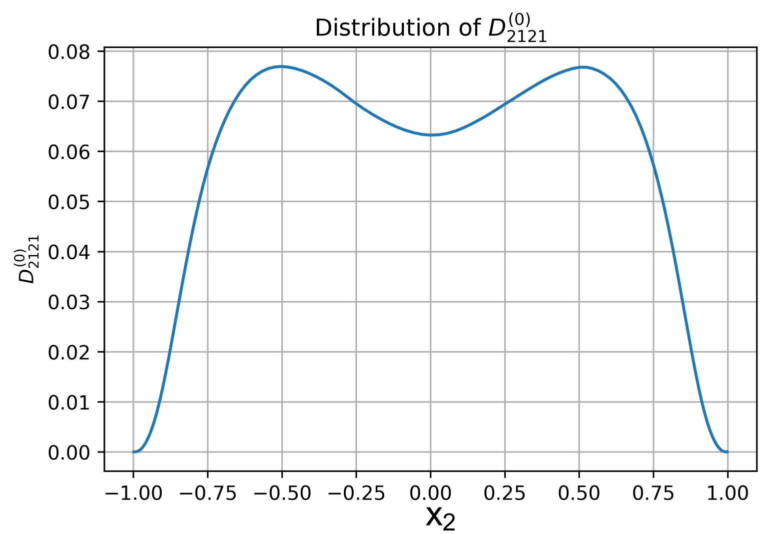
- Only 9 DNSs required to compute D^0_{jilk}
- All 81 coefficients computed versus distance from wall
  - This is the eddy-diffusivity tensor !
    - Represents truncated operator up to the leading (local) term

$$-\overline{u_j' v_i'} = \mathcal{D}\nabla\overline{\mathbf{v}} = \int_{\mathbf{y}} D_{jilk}\left(\mathbf{x},\mathbf{y}\right)\frac{\partial \overline{v_k}}{\partial x_l}|_{\mathbf{y}}\mathbf{dy}.$$

    - Analysis of non-local terms in progress

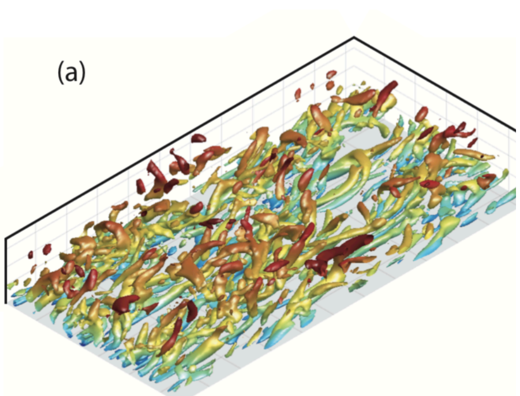
  - Example profile (one out of 81 profiles)

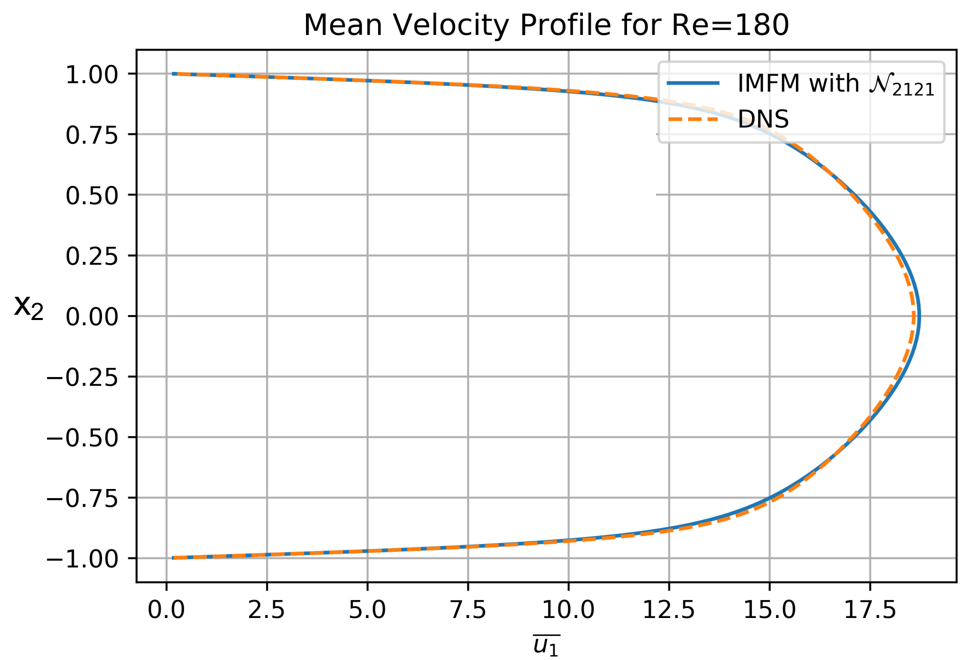
  - MFM can measure eddy diffusivity on the centerline !



IMFM Applied to Turbulent Channel Flow at Re_τ=180 (2/3)

- Leading order model (D^0) is sufficient for RANS prediction of channel flow (nonlocal effects not dominant for this example)





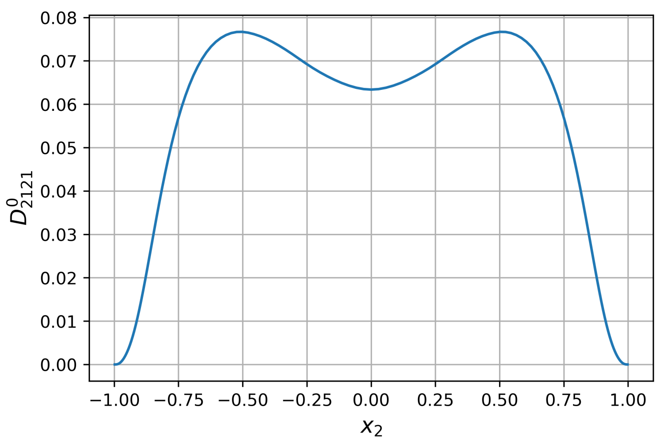
- Error due to neglecting of non-local effects ~ 1.5% (converged)

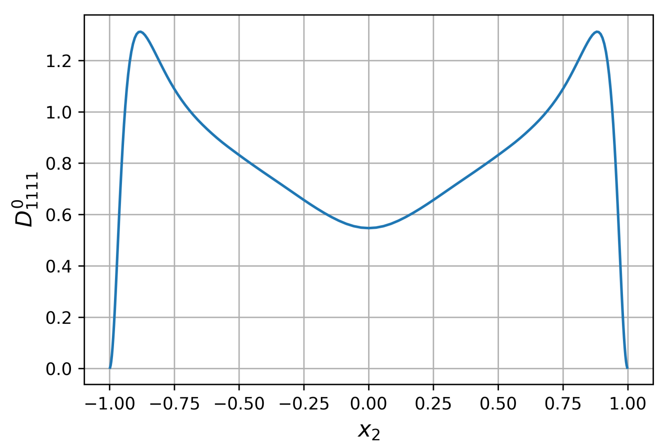
IMFM Applied to Turbulent Channel Flow at Re_τ=180 (3/3)

- Eddy diffusivity is highly anisotropic (non-Boussinesq)
  - Let's examine streamwise momentum equation

$$\frac{\partial \overline{v_1}}{\partial t} + \ldots = \ldots + \frac{\partial}{\partial x_2}\left(D^0_{2121}\frac{\partial \overline{v_1}}{\partial x_2}\right) + \ldots. + \frac{\partial}{\partial x_1}\left(D^0_{1111}\frac{\partial \overline{v_1}}{\partial x_1}\right)$$

  - Boussinesq approx. prescribes $D^0_{1111}=D^0_{2121}$
  - Our measurement shows D^0_{1111} is ~16 times larger than D^0_{2121}!





  - In parallel flows (e.g. channel) the difference does not matter
  - What about onset of separation?

Final Words

- **We developed a rheometer for turbulent flows**

- Standard rheometer for laminar flow measures momentum diffusivity
  - Assumes the underlying Brownian motion (transporter of momentum) remains unaffected by rheometry
  - MFM honors this condition for turbulent flows

transporter of momentum

momentum

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_j u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + r_i,$$

  - **MFM informs closure model forms:**
    → anisotropy, non-locality

› **Ref:** Mani, A. and Park, D., "Macroscopic forcing method: a tool for turbulence modeling and analysis of closures," Physical Review Fluids

Thank you